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On a multiscale continuous percolation model with unbounded defects

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Abstract. We study the multiscale (fractal) percolation in dimension greater than or equal to 2, where the model at each level is the Poisson Boolean model $[\![\lambda, \rho]\!]$. Also, the random radius ρ is supposed to be unbounded. We prove that if the rate λ of Poisson field is less than some critical value, then by choosing the scaling parameter large enough one can assure that there is no multiscale percolation. Another result of this paper is that if the expectation of $\rho^{2\alpha d}$ is finite, then the expectation of the size of the cluster raised to the power α is also finite for small λ, which is a generalization of one of the results of [8].

Keywords: multiscale percolation, Poisson Boolean model.

Mathematical subject classification: 60K35, 60G60.

1 Introduction and results

In this paper we study the multiscale percolation of unbounded Poisson Boolean models in the dimension $d \geq 2$.

The Poisson Boolean model is probably the most famous example of continuum percolation models. It may be described in the following way: First, take a realization of the Poisson field with rate $\lambda > 0$ in $\mathbb{R}^d$. Then, into each point of the field put a ball of random radius ρ independently of everything. The object of interest is the union of all those balls. Models with balls substituted by defects of arbitrary shapes also were studied (cf. [8, 16, 21, 22]). A complete review of the subject can be found in [11]; cf. also in [1, 6, 19, 20] some recent developments.

The main goal that we pursue in this paper is to obtain an extension of the known results for bounded models to the case of unbounded models (which is usually

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rather non-trivial task in the percolation theory). Let us explain this in more detail. For the Poisson Boolean model, the case which is best studied is when $\rho < const$ a.s. When ρ is unbounded, some "strange" situations, which contradict to the discrete percolation intuition, are possible. For example (cf. [8]), if $\mathbf{E}\rho^{2d-1} < \infty$ and $\mathbf{E}\rho^{2d} = \infty$, then for λ small enough there are no infinite clusters almost surely, while the expected size of the cluster is infinite, i.e. the critical points do not coincide. Even if ρ, for instance, has exponential tail, the classical percolation results, such as coincidence of critical points, exponential decay of the size of the cluster in the subcritical phase etc. are unknown. In particular, the quantity λ_ρ (defined below) may be different from the "classical" critical points, such as, for example, the critical rate λ_{cr} which separates percolation from the absence of percolation. Therefore, Theorem 1.2 below (which is a generalization of one of the results of [8]) is of independent interest.

In percolation theory some interest was attracted by the problems which arise when the percolation model is formed by the following procedure. Some random set is *rescaled* (probably, more than once), and the result is in some sense superpositioned with the independent copy of the original random set. First such model was introduced by Mandelbrot [10], and extensively studied later on, cf., for example, [2, 3, 4, 13, 14, 15, 17]. Continuum models of such kind also attracted some attention, cf. [5, 11, 12, 14, 18, 20]. All the papers cited above study models with bounded defects; here we consider the situation when the model of each level contains the defects of arbitrarily large size. The main goal of this paper is to obtain a generalization of Theorem 1.1 of [14] to the case os unbounded defects. As noted above, in this case the study of percolation models is indeed much more difficult, and in fact the attempt to use the method of [14] straightforwardly fails.

Now, let us describe the model of interest. Consider Poisson Boolean model $\mathcal{M}_0 = [\![\lambda, \rho]\!]$ (see in [11] the definitions and some general theory), where $\lambda > 0$ is the rate of Poisson field and $\rho > 0$ is the random radius. Here and in the sequel double square brackets $[\![\cdot, \cdot]\!]$ stand for a Poisson Boolean model. Also, $\mathcal{U}[\![\cdot, \cdot]\!] \subset \mathbb{R}^d$ denotes the union of all balls with positive radius in the model, and $\mathcal{X}[\![\cdot, \cdot]\!]$ denotes the field of the centers of those balls. We construct the multiscale Poisson Boolean model in $\mathbb{R}^d$ in the following way. Fix $R > 1$. Level-i model is $\mathcal{M}_i = [\![\lambda R^{id}, \rho R^{-i}]\!]$, where the Poisson point process and the radii of the balls are independent of what happens on all other levels. The balls from $\mathcal{M}_i$ are called *level-i balls*. Denote $U^{(i)} = \mathcal{U}(\mathcal{M}_i)$. The object of interest is the random set $U = \cup_{i=0}^{\infty} U^{(i)}$. Say that in this model percolation occurs if almost surely there exists a continuous path $\gamma : \mathbb{R} \mapsto U$, such that γ is not contained in any finite box.

Denote

$$\begin{aligned} \varphi(n) &= \mathbf{P}\{\|U^{(0)}(0)\| > n\}, \text{ and} \\ \Theta(\rho) &= \{\lambda > 0 : \varphi(n) = o(n^{-d}), n \to \infty\} \subset \mathbb{R}_+, \end{aligned}$$

where $U^{(0)}(0)$ denotes the connected component from $U^{(0)}$ which contains 0 and $\|U^{(0)}(0)\|$ denotes its diameter.

We suppose that the following conditions are satisfied:

Condition A. The set $\Theta(\rho)$ is not empty.

Let

$$\lambda_\rho = \{\sup \lambda : \lambda \in \Theta(\rho)\}.$$

Clearly, Condition A assures that $\lambda_\rho > 0$.

Condition B. The random radius ρ satisfies

$$\lim_{R\to\infty} \sup_{x\geq 1/2} \frac{R^d \hat{F}_\rho(xR)}{\hat{F}_\rho(x)} = 0, \tag{1}$$

with the convention $0/0 = 0$, where $\hat{F}_\rho(x) = \mathbf{P}\{\rho \geq x\}$.

Besides supposing that ρ has the tail which decreases rapidly enough, Condition B also requires some regularity of the distribution of ρ. But this requirement is not very stringent, for example, if

- there exist $\gamma_1, \gamma_2 > 0$ such that

$$\exp(-\gamma_1 x) \leq \hat{F}_\rho(x) \leq \exp(-\gamma_2 x), \text{ or}$$

- there exist $C_1, C_2 > 0$, $\gamma > d$ such that

$$C_1 x^{-\gamma} \leq \hat{F}_\rho(x) \leq C_2 x^{-\gamma}, \text{ or}$$

- ρ has Poisson distribution,

then Condition B is satisfied. Note that any bounded random variable ρ also satisfies Condition B.

Our main result is the following

Theorem 1.1. *If Conditions A, B are satisfied, then for any $\lambda < \lambda_\rho$ there exists $R_0 = R_0(\lambda)$ such that for all $R \geq R_0$ there is no percolation in the set U.*

Note that the set $\mathbb{R}^d \setminus U$ has the Lebesgue measure 0, but Theorem 1.1 shows that it may be the case that there is no percolation in the set U.

Remark. Suppose that $\rho = 1$ a.s. Then, we have the results about the coincidence of the critical points and the exponential decay of the size of the cluster (cf. [16]), so Condition A holds and $\lambda_\rho = \lambda_{cr}$. As noted above, Condition B also holds in this case. Thus, Theorem 1.1 indeed generalizes Theorem 1.1 of [14].

It is important to mention that the verification of Condition A may be very difficult, because it involves the properties of the whole cluster, not just the single radius distribution ρ. For example, one of the results of [8] implies that if $\mathbf{E}\rho = \infty$, then $\|U^{(0)}(0)\| = \infty$ a.s. for any $\lambda > 0$. Nevertheless, note that if $\mathbf{E}\|U^{(0)}(0)\|^{d+1} < \infty$, then Chebyshev inequality implies that

$$\mathbf{P}\{\|U^{(0)}(0)\| > n\} = O(n^{-(d+1)}),$$

and so Condition A is satisfied. Then, we prove the following result, which may be of independent interest. It is a generalization of one of the results of [8] where the case $\alpha = 1$ was considered.

Theorem 1.2. *If* $\mathbf{E}\rho^{2\alpha d} < \infty$ *for some* $\alpha \in \mathbb{N}$, *then there exists* $\lambda_0 > 0$ *such that* $\mathbf{E}\|U^{(0)}(0)\|^{\alpha} < \infty$ *for all* $\lambda < \lambda_0$.

Thus, for Condition A to hold, Theorem 1.2 implies that it is *sufficient* that $\mathbf{E}\rho^{2d(d+1)} < \infty$.

2 Proof of Theorem 1.1

Denote $U_n = \cup_{i=0}^{n} U^{(i)}$. Let $S_m \subset \mathbb{R}^d$ be the sphere of radius m centered at 0. Following [14], to prove the absence of percolation in U it is sufficient to prove that the sets U_n are in the subcritical phase uniformly in n, i.e.

$$\mathbf{P}\{\text{there exists a path connecting 0 to } S_m \text{ in } U_n\} < \varepsilon_{m,n}, \tag{2}$$

where $\varepsilon_{m,n} \to 0$ uniformly in n as $m \to \infty$. Fix some n and consider the percolation problem in U_n.

Definition 2.1. *We say that one Poisson Boolean model* $[\![\lambda_1, \rho_1]\!]$ *is dominated by another Poisson Boolean model* $[\![\lambda_2, \rho_2]\!]$ *when it is possible to couple them in such a way that* $\mathcal{U}[\![\lambda_1, \rho_1]\!] \subset \mathcal{U}[\![\lambda_2, \rho_2]\!]$.

We need the following

Lemma 2.1. *Let* $[\![\lambda_1, \rho_1]\!]$ *and* $[\![\lambda_2, \rho_2]\!]$ *be two Poisson Boolean models. If* ρ_1, ρ_2 *are such that*

$$\lambda_1 \hat{F}_{\rho_1}(x) \le \lambda_2 \hat{F}_{\rho_2}(x) \tag{3}$$

for all $x > 0$, *than* $[\![\lambda_1, \rho_1]\!]$ *is dominated by* $[\![\lambda_2, \rho_2]\!]$.

Proof. Note that we can make $\lambda_1 = \lambda_2$ by enlarging the smaller of the lambdas and adding the positive mass in 0 to the respective ρ. It can be easily seen that this does not affect the validity of (3) (if λ is enlarged up to λ^*, then for $x > 0$ $\hat{F}(x)$ is substituted by $\hat{F}^*(x) = \frac{\lambda}{\lambda^*}\hat{F}(x)$, so that $\lambda\hat{F}(x) = \lambda^*\hat{F}^*(x)$). So, without loss of generality, suppose that $\lambda_1 = \lambda_2 =: \lambda^*$, and thus

$$\hat{F}_{\rho_1}(x) \le \hat{F}_{\rho_2}(x) \tag{4}$$

for all $x > 0$. The rest of the proof is quite standard. Having the configuration $x_1, x_2, \dots$ of Poisson point process with rate λ^* consider the sequence of independent random variables ζ_i, $i = 1, 2, \dots$, uniformly distributed in $[0, 1]$. Let $F_{\rho_j} = 1 - \hat{F}_{\rho_j}$ denote the distribution function of ρ_j, $j = 1, 2$. We define the coupling of the two models by $\rho_j(i) = F_{\rho_j}^{-1}(\zeta_i)$, where $\rho_j(i)$ is the realization of the random variable ρ_j at the point x_i, $j = 1, 2$. Now, if $\rho_1(i) > \rho_2(i)$, then there exists $y > 0$ such that $F_{\rho_1}^{-1}(\zeta_i) > y > F_{\rho_2}^{-1}(\zeta_i)$, that is, $F_{\rho_1}(y) < \zeta_i < F_{\rho_2}(y)$ which contradicts (4). Thus, the proof of Lemma 2.1 is completed. □

For $L \ge 0$ denote $\rho^{\ge L} = \rho\mathbf{1}\{\rho \ge L\}$ and $\rho^{<L} = \rho - \rho^{\ge L}$. Note that without loss of generality one can suppose that there exists $a > 0$ such that $\rho \in \{0\} \cup [a, +\infty)$ a.s. To show that, first, let us prove that $[\![\lambda, \rho]\!]$ is dominated by

$$\left[\!\!\left[\lambda\left(1 + \frac{\mathbf{P}\{\rho < a\}}{\mathbf{P}\{\rho \ge a\}}\right), \rho^{\ge a}\right]\!\!\right].$$

Indeed, by Lemma 2.1 it is sufficient to prove that

$$\lambda\hat{F}_\rho(x) \le \lambda\left(1 + \frac{\mathbf{P}\{\rho < a\}}{\mathbf{P}\{\rho \ge a\}}\right)\hat{F}_{\rho^{\ge a}}(x) \tag{5}$$

for all $x > 0$. As

$$\hat{F}_{\rho^{\ge a}}(x) = \begin{cases} \hat{F}_\rho(x), & \text{if } x \ge a, \\ \hat{F}_\rho(a), & \text{if } 0 < x < a, \end{cases} \tag{6}$$

the inequality (5) trivially holds for $x \ge a$. For $0 < x < a$ we have $\mathbf{P}\{\rho \ge a\}\hat{F}_\rho(x) \le \hat{F}_\rho(a)$, which is equivalent to

$$\lambda\hat{F}_\rho(x) \le \lambda\left(1 + \frac{\mathbf{P}\{\rho < a\}}{\mathbf{P}\{\rho \ge a\}}\right)\hat{F}_\rho(a),$$

which proves (5). Now, as $\lambda < \lambda_\rho$, choosing a small, one can make $\mathbf{P}\{\rho < a\}$ small enough to assure that

$$\lambda\left(1 + \frac{\mathbf{P}\{\rho < a\}}{\mathbf{P}\{\rho \ge a\}}\right) \in \Theta(\rho) \subset \Theta(\rho^{\ge a}).$$

So one can consider $\rho^{\geq a}$ instead of ρ; it means that the radii of all the nontrivial balls in $[\![\lambda, \rho]\!]$ are greater or equal to a.

Choose $\varepsilon > 0$, $0 < \alpha < 1$ such that $\lambda + \varepsilon \in \Theta((1+\alpha)\rho)$ (by the rescaling argument, it is equivalent to $(\lambda+\varepsilon)(1+\alpha)^d \in \Theta(\rho)$) and fix some $\alpha_0 < \alpha$, $\varepsilon_0 < \varepsilon$.

Lemma 2.2. *If Condition B is satisfied, then there exists $R_{\varepsilon_0,\alpha_0}$ such that for any $R > R_{\varepsilon_0,\alpha_0}$ the union of two independent Boolean models $\mathcal{M}_i = [\![\lambda R^{id}, \rho R^{-i}]\!]$ and $[\![(\lambda+\varepsilon_0)R^{(i+1)d}, ((1+\alpha_0)\rho)^{\geq R} R^{-(i+1)}]\!]$ is dominated by $[\![(\lambda+\varepsilon_0)R^{id}, (1+\alpha_0)\rho R^{-i}]\!]$, $i = 0, \dots, n-1$.*

Proof. Note that Condition B implies that

$$\lim_{R\to\infty} \sup_{x>0} \frac{R^d \hat{F}_{\rho^{\geq R/2}}(xR)}{\hat{F}_\rho(x)} = 0. \tag{7}$$

Indeed, as

$$\hat{F}_{\rho^{\geq R/2}}(xR) = \begin{cases} \hat{F}_\rho(xR), & \text{if } x \geq 1/2, \\ \hat{F}_\rho(R/2), & \text{if } 0 < x < 1/2 \end{cases} \tag{8}$$

we have

$$\sup_{x>0} \frac{R^d \hat{F}_{\rho^{\geq R/2}}(xR)}{\hat{F}_\rho(x)} = \sup_{x\geq 1/2} \frac{R^d \hat{F}_\rho(xR)}{\hat{F}_\rho(x)}.$$

Fix $i \in \{0, 1, \dots, n-1\}$. Denote $\eta_0 := ((1+\alpha_0)\rho)^{\geq R}$ and $\eta_1 := (1+\alpha_0)\rho$. Note that the model $[\![(\lambda+\varepsilon_0)R^{id}, (1+\alpha_0)\rho R^{-i}]\!]$ can be represented as the union of two Poisson Boolean models: $[\![\lambda R^{id}, (1+\alpha_0)\rho R^{-i}]\!]$ and $[\![\varepsilon_0 R^{id}, (1+\alpha_0)\rho R^{-i}]\!]$. Clearly, $\mathcal{M}_i$ is dominated by the first one, so it remains to prove that $[\![(\lambda+\varepsilon_0)R^{(i+1)d}, ((1+\alpha_0)\rho)^{\geq R} R^{-(i+1)}]\!]$ is dominated by $[\![\varepsilon_0 R^{id}, (1+\alpha_0)\rho R^{-i}]\!]$. By Lemma 2.1 we need to prove that for all R large enough

$$(\lambda+\varepsilon_0)R^{(i+1)d} \hat{F}_{\eta_0 R^{-(i+1)}}(x) \leq \varepsilon_0 R^{id} \hat{F}_{\eta_1 R^{-i}}(x),$$

for all $x > 0$. For this, it is sufficient to prove that

$$\frac{R^d \hat{F}_{\eta_0 R^{-(i+1)}}(x)}{\hat{F}_{\eta_1 R^{-i}}(x)} \to 0$$

uniformly in x, as $R \to \infty$.

Due to (7) and the fact that $\hat{F}_{\rho^{\geq R/2}} \geq \hat{F}_{\rho^{\geq R/(1+\alpha_0)}} = \hat{F}_{((1+\alpha_0)\rho)^{\geq R}}$ if $\alpha_0 < 1$, one has

$$\frac{R^d \hat{F}_{\eta_0 R^{-(i+1)}}(x)}{\hat{F}_{\eta_1 R^{-i}}(x)} = \frac{R^d \hat{F}_{\eta_0}(R^{i+1}x)}{\hat{F}_{\eta_1}(R^i x)} \to 0$$

uniformly in x, as $R \to \infty$. Lemma 2.2 is proved. □

Take $R > R_{\varepsilon_0,\alpha_0}$. First, split $[\![\lambda R^{nd}, \rho R^{-n}]\!]$ into two independent models $[\![\lambda R^{nd}, \rho^{<R} R^{-n}]\!]$ and $[\![\lambda R^{nd}, \rho^{\geq R} R^{-n}]\!]$. We apply the Lemma 2.2 to

$$[\![\lambda R^{(n-1)d}, \rho R^{-(n-1)}]\!] \cup [\![\lambda R^{nd}, \rho^{\geq R} R^{-n}]\!]$$

and obtain that it is dominated by

$$[\![(\lambda + \varepsilon_0) R^{(n-1)d}, (1+\alpha_0)\rho R^{-(n-1)}]\!].$$

So, the model

$$[\![\lambda R^{(n-1)d}, \rho R^{-(n-1)}]\!] \cup [\![\lambda R^{nd}, \rho R^{-n}]\!]$$

is dominated by

$$[\![(\lambda + \varepsilon_0) R^{(n-1)d}, (1+\alpha_0)\rho R^{-(n-1)}]\!] \cup [\![\lambda R^{nd}, \rho^{<R} R^{-n}]\!].$$

Obviously, $[\![\lambda R^{nd}, \rho^{<R} R^{-n}]\!]$ is dominated by $[\![(\lambda+\varepsilon_0) R^{nd}, ((1+\alpha_0)\rho)^{<R} R^{-n}]\!]$. In the same way we obtain that

$$[\![\lambda R^{(n-2)d}, \rho R^{-(n-2)}]\!] \cup [\![(\lambda + \varepsilon_0) R^{(n-1)d}, (1+\alpha_0)\rho R^{-(n-1)}]\!]$$

is dominated by

$$[\![(\lambda + \varepsilon_0) R^{(n-2)d}, (1+\alpha_0)\rho R^{-(n-2)}]\!] \cup [\![(\lambda + \varepsilon_0) R^{(n-1)d}, ((1+\alpha_0)\rho)^{<R} R^{-(n-1)}]\!],$$

and so on. Thus we get that the union of independent Poisson Boolean models

$$\bigcup_{i=0}^{n} \mathcal{M}_i = \bigcup_{i=0}^{n} [\![\lambda R^{id}, \rho R^{-i}]\!]$$

is dominated by

$$[\![(\lambda + \varepsilon_0), (1+\alpha_0)\rho]\!] \cup \left(\bigcup_{i=1}^{n} [\![(\lambda + \varepsilon_0) R^{id}, ((1+\alpha_0)\rho)^{<R} R^{-i}]\!] \right),$$

where all the $n+1$ Poisson Boolean models are independent as well. Abbreviate $D^{(i)} := \mathcal{U}[\![(\lambda+\varepsilon_0)R^{id}, ((1+\alpha_0)\rho)^{<R}R^{-i}]\!]$, $\mathcal{X}_i := \mathcal{X}[\![(\lambda+\varepsilon_0)R^{id}, ((1+\alpha_0)\rho)^{<R}R^{-i}]\!]$, $i = 1, \dots, n$, and $D^{(0)} := \mathcal{U}[\![(\lambda+\varepsilon_0), (1+\alpha_0)\rho]\!]$, $\mathcal{X}_0 := \mathcal{X}[\![(\lambda+\varepsilon_0), (1+\alpha_0)\rho]\!]$. Remember that $\mathcal{X}[\![\cdot,\cdot]\!]$ denotes the set of Poisson points which carry a ball of positive radius in the respective model. Take $\beta > 0$ such that $(1+\alpha_0)(1+\beta) < 1+\alpha$, so

$$\lambda + \varepsilon \in \Theta((1+\alpha_0)(1+\beta)\rho). \tag{9}$$

Denote $W^{(i)} := \mathcal{U}[\![(\lambda+\varepsilon)R^{id}, (1+\beta)((1+\alpha_0)\rho))^{<R}R^{-i}]\!]$, $i = 1, \dots, n$ and $W^{(0)} := \mathcal{U}[\![(\lambda+\varepsilon), (1+\beta)(1+\alpha_0)\rho]\!]$. Also, let $V^{(i)} = \mathcal{U}[\![(\lambda+\varepsilon_0)R^{id}, (1+\beta)((1+\alpha_0)\rho))^{<R}R^{-i}]\!]$, $i = 1, \dots, n$ and $V^{(0)} := \mathcal{U}[\![(\lambda+\varepsilon_0), (1+\beta)(1+\alpha_0)\rho]\!]$, where $V^{(j)}$ uses the same Poisson points process $\mathcal{X}_j$ as $D^{(j)}$, $j = 0, 1, \dots, n$. Note that V_j is in fact D_j expanded by the factor $(1+\beta)$, $j = 0, 1, \dots, n$, and V_j-s are independent.

For $i = 0, \dots, n-1$ consider a partition of the space into the cubes with the edge length $aR^{-i}/\sqrt{d}$, which we call level-i cubes. Note that the size of the cube is chosen in such a way that if there is a center of a ball from $D^{(i)}$ inside the cube, then the latter is completely covered by the ball.

From this point on, we use some ideas of [14, 15].

We define now *passable sets* $P_0, \dots, P_{n-1}$, and *good sets* $G_0, \dots, G_n$.

Definition 2.2. *First, the good level-n set is defined by $G_n := D^{(n)}$. For the level $i < n$, we say that level-i cube is passable if it has nonempty intersection with some connected component of diameter greater than $2a\beta R^{-i}$ of the good level-$(i+1)$ set G_{i+1}. The passable level-i set P_i is defined as the union of all the passable level-i cubes. The good level-i set G_i is defined by $G_i := P_i \cup V^{(i)}$.*

Lemma 2.3. *Percolation in U_n implies percolation in G_0.*

Proof. As we have seen, U_n is dominated by $D^{(n)} \cup \dots \cup D^{(0)}$. Using this, one gets that, to prove Lemma 2.3, it is enough to prove the following fact:

$$\Big(\text{percolation in } \bigcup_{j=0}^{n} D^{(j)}\Big) \Rightarrow \Big(\text{percolation in } G_k \cup \bigcup_{j=0}^{k-1} V^{(j)}\Big) \tag{10}$$

for $k = n, n-1, \dots, 0$ (indeed, take $k = 0$ in (10) to obtain the statement of Lemma 2.3).

We prove (10) by induction. First, for $k = n$, we have that $G_n = D^{(n)}$, so (10) follows. Now, suppose that (10) holds for $k+1$, let us prove it for k, i.e. let us prove that

$$\left(\text{percolation in } G_{k+1} \cup \bigcup_{j=0}^{k} V^{(j)}\right) \Rightarrow \left(\text{percolation in } G_k \cup \bigcup_{j=0}^{k-1} V^{(j)}\right). \qquad (11)$$

Take any infinite continuous path γ in $G_{k+1} \cup V^{(k)} \cup \cdots \cup V^{(0)}$. Consider $(G_{k+1} \setminus (D^{(0)} \cup \cdots \cup D^{(k)})) \cap \gamma$; it can be decomposed into finite or countable number of connected segments $\gamma_1, \gamma_2, \ldots$ Take any γ' from that collection, suppose that the two extremal points of it belong to $D^{(i_1)}$ and $D^{(i_2)}$, where $i_1, i_2 \leq k$. There are two possibilities:

- diameter of γ' is less than $2a\beta R^{-k}$; in this case $\gamma' \subset V^{(i_1)} \cup V^{(i_2)}$;
- diameter of γ' is greater or equal to $2a\beta R^{-k}$; in this case γ' is covered by passable level-k cubes, so $\gamma' \subset P_k$.

In both cases we have

$$\gamma' \subset P_k \cup V^{(k)} \cup \cdots \cup V^{(0)} = G_k \cup V^{(k-1)} \cup \cdots \cup V^{(0)},$$

which gives the proof of (11), and, consequently, of (10) and Lemma 2.3. □

One of the main ingredients of the proof of Theorem 1.1 is the following

Proposition 2.1. *If for fixed $\varepsilon_0, \alpha_0, \beta$ the scaling parameter R is large enough, then G_i can be dominated by $W^{(i)}$, $i = 0, \ldots, n$.*

Proof. Let us prove the proposition by induction. Clearly, by Definition 2.2, the set $G_n = D^{(n)}$ can be dominated by $W^{(n)}$. Suppose that the proposition holds for level $k+1$; let us prove it for level k.

Fix some level-k cube K. Denote $\delta = aR^{-(k+1)}/\sqrt{d}$. Let

$$K_\delta = \{x \in R^d : \operatorname{dist}(x, K) \leq \delta\}$$

and choose some δ-net $N^{(\delta)}(K_\delta)$ in K_δ. Note that it is possible to choose this δ-net in such a way that $\operatorname{card}(N^{(\delta)}(K_\delta))$ is proportional to R^d, where $\operatorname{card}(A)$ stands for the cardinality of the set A. We have

$$\begin{aligned}
&\mathbf{P}\{K \text{ is passable}\} \\
&\quad \leq \mathbf{P}\{\text{there exists } x \in N^{(\delta)}(K_\delta) \text{ which belongs to some connected} \\
&\qquad\quad \text{component of diameter greater than } 2a\beta R^{-k} \text{ of } G_{k+1}\} \qquad (12) \\
&\quad \leq \operatorname{card}(N^{(\delta)}(K_\delta))\mathbf{P}\{\|U^{(k+1)}(0)\| > 2a\beta R^{-k}\}
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{card}(N^{(\delta)}(K_\delta))\mathbf{P}\{\|U^{(0)}(0)\| > 2a\beta R\} \\
&\le cR^d\varphi(R) =: \psi(R) = R^d o(R^{-d}) = o(1), \quad R \to \infty.
\end{aligned}
$$

Note that, as

- we are interested (in the definition of passable cubes) in the connected components of G_{k+1} with diameter greater than $2a\beta R^{-k}$, and
- the balls from $V^{(k+1)}$ have radius less than $(1+\beta)R^{-k}$,

denoting $b := 2\max\{\lceil 2\beta\sqrt{d}\rceil, \lceil 2a^{-1}(1+\beta)\sqrt{d}\rceil\}$ (recall that the edge of level-k cube is equal to $aR^{-k}/\sqrt{d}$), where $\lceil x\rceil$ denotes the smallest integer greater or equal than x, we have that if some two level-k cubes have at least b cubes between them, then those two cubes are passable or not independently. So, when $\psi(R)$ is small enough, by the result of [9] we get that the random field of passable level-k cubes can be dominated by Bernoulli random field with parameter $\sigma(R)$, and this parameter can be made arbitrarily close to 0 by choosing R large enough. Note that the choice of R depends only on $d, \lambda, \varepsilon_0, \alpha_0$, but not on n.

Now, in its turn the Bernoulli random field with parameter $\sigma(R)$ of level-k cubes can be dominated by the field of balls of radius aR^{-k}, centers of which form Poisson field in $\mathbb{R}^d$ with rate $\varepsilon' R^{kd}$, and ε' can be made arbitrary close to 0 by choosing R. To justify this, we consider the following coupling between the above two fields. If, given a realization of the Poisson fields of centers of the balls, a given cube contains at least one point of the Poisson field, then the cube is selected. Clearly, the states of the cubes are independent and

$$
\sigma(R) = \mathbf{P}\{\text{the cube is selected}\} = 1 - \exp\Big(\varepsilon'\Big(\frac{a}{\sqrt{d}}\Big)^d\Big). \tag{13}
$$

Note that if the cube contains a center of the ball, then, as noted before, the cube is completely covered by the ball, so the field of the cubes is indeed dominated by the field of the balls. Since by choosing R large we can made $\sigma(R)$ arbitrarily close to 0, we have that ε' determined by (13) will be arbitrarily close to 0 as well. Take R such that $\varepsilon' < \varepsilon - \varepsilon_0$.

Thus, the good level-k set G_k is dominated by

$$
V^{(k)} \cup [\![\varepsilon' R^{kd}, aR^{-k}]\!].
$$

As $\varepsilon' < \varepsilon - \varepsilon_0$ we have that $V^{(k)} \cup [\![\varepsilon' R^{kd}, aR^{-k}]\!]$ is dominated by $W^{(k)}$. The proof of Proposition 2.1 is completed. □

The proof of Theorem 1.1 is now straightforward. By Lemma 2.3,

$$
\Big(\text{no percolation in } G_0\Big) \Rightarrow \Big(\text{no percolation in } U_n\Big).
$$

By the choice of ε, α, α_0 and Proposition 2.1, the set G_0 (and therefore U_n) is in the subcritical phase uniformly in n. Thus, Theorem 1.1 is proved. □

3 Proof of Theorem 1.2

First, note that it is sufficient to prove the theorem only in the case when ρ takes only positive integer values. Indeed, if ρ takes values other than positive integers, then consider the model where ρ is substituted by $\rho_{int} := \lceil \rho \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater or equal than x. As $\rho_{int} \leq \rho + 1$, it is straightforward to get that if $\mathbf{E}\rho^{2\alpha d} < \infty$, then $\mathbf{E}\rho_{int}^{2\alpha d} < \infty$. As $\rho \leq \rho_{int}$, a simple coupling argument applies.

For $j = 1, 2, \dots$ denote $p_j = \mathbf{P}\{\rho = j\}$. If there is a ball of radius i, the number of balls of radius j which have nonempty intersection with it (we denote this number by $\eta^{(i,j)}$) has Poisson distribution with mean $\psi^{(i,j)} := \lambda \pi_d (i+j)^d p_j$ (cf. [8]), where π_d is the volume of d-dimensional unit ball. Now, following [8], we are going to construct a multitype branching process $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2, \dots$, which majorizes the percolation process. Here $\mathcal{Z}_n = (Z_n^1, Z_n^2, Z_n^3, \dots)$, where Z_n^j is the number of particles of type j (i.e., balls of radius j) in the n-th generation, and $\mathcal{Z}_0 = e_i$, where $e_i = (0, \dots, 0, 1, 0 \dots)$, with 1 on the i-th place. The dynamics of the branching process is described as follows: each particle of type i is substituted by $\eta^{(i,j)}$ particles of type j independently of all the other particles, and the random variables $\eta^{(i,j)}$, $j = 1, 2, 3, \dots$ are *independent* and have Poisson distribution with mean $\psi^{(i,j)}$.

As in [8], we have

$$\mu_{i,j} := \mathbf{E}\eta^{(i,j)} = \psi^{(i,j)} \leq \tilde{C}\lambda i^d j^d p_j \tag{14}$$

for $\tilde{C} = 2^d \pi_d$. We are going to use the following simple fact: if η has Poisson distribution with mean ψ, then (see [7], Section 1.3)

$$\mathbf{E}\eta^k = \sum_{j=1}^{k} B_{j,k} \psi^j \tag{15}$$

for some positive constants $B_{j,k}$, $j = 1, \dots, k$, $k = 1, 2 \dots$ If $\lambda \leq 1$, using (14) and (15) we get

$$\mathbf{E}(\eta^{(i,j)})^k \leq C\lambda i^{kd} j^{kd} p_j \tag{16}$$

for some positive constant $C = C(\alpha)$ for all $k \leq \alpha$.

Let us introduce some notation. In the course of the proof of this theorem, we will often need to deal with collections of positive integers, where those positive integers are not necessarily distinct. It is then natural to group the equal numbers together, thus representing the collection as

$$(\mathbf{t}, \mathbf{w}, h) = (t_1, w_1; \dots ; t_h, w_h; h) \in \mathbb{N}^{2h+1},$$

where all t_i-s are different, w_i can be viewed as the number of repetitions of t_i in the collection, and h is the number of distinct elements in the collection. Now, given the collection $(\mathbf{t}, \mathbf{w}, h)$, denote

$$\begin{aligned}
\varphi(\mathbf{t}, \mathbf{w}, h) &= t_1^{w_1 d} \cdots t_h^{w_h d}, \\
\Phi(\mathbf{t}, \mathbf{w}, h) &= t_1^{w_1 d} \cdots t_h^{w_h d} p_{t_1} \cdots p_{t_h}, \\
Z_i(\mathbf{t}, \mathbf{w}, h) &= (Z_i^{t_1})^{w_1} \cdots (Z_i^{t_h})^{w_h}, \\
\mu_{i,\,(\mathbf{t},\mathbf{w},h)}^{(n)} &= \mathbf{E}(Z_n(\mathbf{t}, \mathbf{w}, h) \mid \mathcal{Z}_0 = e_i).
\end{aligned}$$

We will write $(\mathbf{t}, \mathbf{w}, h; \beta)$ when it is necessary to keep track of the total number of elements $\beta = \sum_{i=1}^h w_i$. Also, let $\mathcal{F}_m$ be the σ-algebra generated by $\mathcal{Z}_0, \mathcal{Z}_1, \dots, \mathcal{Z}_m$.

Lemma 3.1. *For λ small enough we have*

$$\mu_{i,\,(\mathbf{l},\mathbf{k},\gamma;\alpha)}^{(n)} \le K^{n-1}(C\lambda)^n i^{\alpha d} \Phi(\mathbf{l}, \mathbf{k}, \gamma; \alpha), \tag{17}$$

where $C > 0$ and $K > 0$ depend only on α.

Proof. We prove the lemma by induction. For $n = 1$, using (16) and the independence of $\eta^{(i,j)}$ for different j, it can be easily seen that

$$\mu_{i,\,(\mathbf{l},\mathbf{k},\gamma;\alpha)}^{(1)} \le C\lambda i^{\alpha d} \Phi(\mathbf{l}, \mathbf{k}, \gamma; \alpha) \tag{18}$$

if $C\lambda \le 1$.

Suppose that the lemma is proved for $n-1$. We have

$$\begin{aligned}
\mu_{i,\,(\mathbf{l},\mathbf{k},\gamma;\alpha)}^{(n)} &= \mathbf{E}\Big(\mathbf{E}((Z_n^{l_1})^{k_1} \mid \mathcal{F}_{n-1}) \cdots (\mathbf{E}((Z_n^{l_\gamma})^{k_\gamma} \mid \mathcal{F}_{n-1})\Big) \\
&= \mathbf{E}(A_1 \cdots A_\gamma),
\end{aligned} \tag{19}$$

where $A_m = \mathbf{E}((Z_n^{l_m})^{k_m} \mid \mathcal{F}_{n-1})$, $m = 1, \dots, \gamma$.

Let us estimate A_m.

$$\begin{aligned}
A_m &= \mathbf{E}(Z_n^{l_m})^{k_m} \mid \mathcal{F}_{n-1}) \\
&= \mathbf{E}(\eta_1^{(1,l_m)} + \cdots + \eta_{Z_{n-1}^1}^{(1,l_m)} + \eta_1^{(2,l_m)} + \cdots + \eta_{Z_{n-1}^2}^{(2,l_m)} + \cdots)^{k_m} \qquad (20) \\
&= \sum_{s_m=1}^{k_m} \sum_{(\mathbf{t}_m,\mathbf{u}_m,s_m;k_m)} \sum_{\substack{\mathbf{j}_1=(j_{1,\beta};\beta=1,\dots,u_{m,1}),\dots,\\ \mathbf{j}_{s_m}=(j_{s_m,\beta};\beta=1,\dots,u_{m,s_m})}} \mathbf{E} \prod_{i=1}^{s_m} \prod_{\beta=1}^{u_{m,i}} \eta_{j_{i,\beta}}^{(t_{m,i},l_m)} \\
&= \sum_{s_m=1}^{k_m} \sum_{(\mathbf{t}_m,\mathbf{u}_m,s_m;k_m)} \sum_{\mathbf{j}_1,\dots,\mathbf{j}_{s_m}} \prod_{i=1}^{s_m} \mathbf{E} \prod_{\beta=1}^{u_{m,i}} \eta_{j_{i,\beta}}^{(t_{m,i},l_m)} \qquad (21)
\end{aligned}$$

$$\leq \quad C\lambda l_m^{k_m d} p_{l_m} \sum_{s_m=1}^{k_m} \sum_{(\mathbf{t}_m,\mathbf{u}_m,s_m;k_m)} \varphi(\mathbf{t},\mathbf{u},s_m;k_m) Z_{n-1}(\mathbf{t},\mathbf{u},s_m;k_m).$$

The last inequality holds because $\mathbf{E}(\eta_{j_i,1}^{(t_i,l_m)} \cdots \eta_{j_i,u_{m,i}}^{(t_i,l_m)}) \leq C\lambda t_i^{u_{m,i}d} l_m^{u_{m,i}d} p_{l_m}$ (as the $\eta_{\cdot,\cdot}^{(t_{m,i},l_m)}$ are Poisson random variables with mean $\lambda\pi_d(t_{m,i}+l_m)^d \leq 2^d\pi_d\lambda t_{m,i}^d l_m^d$ and with possible repetitions), and the number of summands in the third sum in (21) is $Z_{n-1}(\mathbf{t},\mathbf{u},s_m;k_m)$.

So,

$$\begin{aligned}
\mathbf{E}(A_1\cdots A_\gamma) \quad &\leq \quad (C\lambda)^\gamma \Phi(\mathbf{l},\mathbf{k},\gamma) \\
&\quad \times \sum_{\substack{j\in\{1,\dots,\gamma\} \\ 1\leq s_j\leq k_j}} \sum_{\substack{(\mathbf{t}_m,\mathbf{u}_m,s_m), \\ 1\leq m\leq\gamma}} \varphi(\mathbf{t}_1,\mathbf{u}_1,s_1)\cdots\varphi(\mathbf{t}_\gamma,\mathbf{u}_\gamma,s_\gamma) \\
&\quad \times \mathbf{E}(Z_{n-1}(\mathbf{t}_1,\mathbf{u}_1,s_1)\cdots Z_{n-1}(\mathbf{t}_\gamma,\mathbf{u}_\gamma,s_\gamma)) \qquad (22) \\
&\leq \quad C\lambda\Phi(\mathbf{l},\mathbf{k},\gamma)M(\alpha) \sum_{(\mathbf{t},\mathbf{w},h;\alpha)} \mu_{i,\,(\mathbf{t},\mathbf{w},h;\alpha)}^{(n-1)} \varphi(\mathbf{t},\mathbf{w},h;\alpha) \qquad (23) \\
&\leq \quad C\lambda\Phi(\mathbf{l},\mathbf{k},\gamma)M(\alpha) \qquad (24) \\
&\quad \times \sum_{(\mathbf{t},\mathbf{w},h;\alpha)} \varphi(\mathbf{t},\mathbf{w},h;\alpha) K^{n-2}(C\lambda)^{n-1} i^{\alpha d}\Phi(\mathbf{t},\mathbf{w},h;\alpha) \\
&= \quad (C\lambda)^n i^{\alpha d}\Phi(\mathbf{l},\mathbf{k},\gamma)M(\alpha)K^{n-2} \sum_{(\mathbf{t},\mathbf{w},h;\alpha)} \Phi(\mathbf{t},2\mathbf{w},h;\alpha).
\end{aligned}$$

To pass from (22) to (23) the terms with the same t-s and Z_{n-1}-s were grouped, and on the passage from (23) to (24) the induction assumption was used. The constant $M(\alpha)$ is defined in the following way. Let $M(\mathbf{t},\mathbf{w},h;\alpha)$ be the number of ways to decompose

$$(\mathbf{t},\mathbf{w},h;\alpha) = (t_1,w_1;\dots;t_h,w_h;h;\alpha)$$

into $(\mathbf{t}_1,\mathbf{u}_1,s_1),\dots,(\mathbf{t}_\gamma,\mathbf{u}_\gamma,s_\gamma)$, where

$$(\mathbf{t}_\theta,\mathbf{u}_\theta,s_\theta) = (t_{\theta,1},u_{\theta,1};\dots;t_{\theta,s_\theta},u_{\theta,s_\theta};s_\theta)$$

such that for any $\beta = 1,\dots,h$ we have

$$\sum_{\theta,\chi:\,t_\beta=t_{\theta,\chi}} u_{\theta,\chi} = w_\beta.$$

Put

$$M(\alpha) = \max_{(\mathbf{t},\mathbf{w},h;\alpha)} M(\mathbf{t},\mathbf{w},h;\alpha).$$

As $M(\mathbf{t}, \mathbf{w}, h; \alpha)$ in fact depends only on $(w_1, \dots, w_h)$ and the number of distinct collections $(w_1, \dots, w_h; \alpha)$ is finite, $M(\alpha)$ is finite as well.

Now, we have

$$\begin{aligned}\sum_{(\mathbf{t},\mathbf{w},h;\alpha)} \Phi(\mathbf{t}, 2\mathbf{w}, h; \alpha) &= \sum_{h=1}^{\alpha} \sum_{\mathbf{w}:\, \sum_{i=1}^{h} w_i=\alpha} \sum_{\mathbf{t}} t_1^{2w_1 d} \cdots t_h^{2w_h d} p_{t_1} \cdots p_{t_h} \\ &= \sum_{h=1}^{\alpha} \sum_{\mathbf{w}:\, \sum_{i=1}^{h} w_i=\alpha} \mathbf{E}\rho^{2w_1 d} \cdots \mathbf{E}\rho^{2w_h d} =: L(\alpha) < \infty,\end{aligned} \tag{25}$$

as $\mathbf{E}\rho^{2\alpha d} < \infty$ (to see this, take the term corresponding to $h = 1$ in (25)). Thus, taking $K = M(\alpha)L(\alpha)$, we complete the proof of Lemma 3.1. □

Now we want to find a way to estimate quantities of the form

$$\mathbf{E}(Z_{n_1}^{j_1} \cdots Z_{n_\alpha}^{j_\alpha}). \tag{26}$$

Grouping equal values of n, let us rewrite the collection $(n_i; i = 1, \dots, \alpha)$ as $(m_1, \gamma_1; \dots; m_s, \gamma_s; s)$, $m_1 < m_2 < \cdots < m_s$. Here γ_i is the number of $n_j = m_i$ in $(n_1, \dots, n_\alpha)$, so $\sum_{i=1}^{s} \gamma_s = \alpha$. Let

$$(j_1, \dots, j_\alpha) = ((\mathbf{j}_1, \mathbf{u}_1, v_1; \gamma_1), \dots, (\mathbf{j}_s, \mathbf{u}_s, v_s; \gamma_s)),$$

where

$$(\mathbf{j}_i, \mathbf{u}_i, v_i; \gamma_i) = (j_{i,1}, u_{i,1}; \dots; j_{i,v_i}, u_{i,v_i}; v_i; \gamma_i).$$

Here $u_{\theta,\chi}$ is the number of terms $Z_{m_\theta}^{j_{\theta,\chi}}$ in (26). So

$$\begin{aligned}\mathbf{E}(Z_{n_1}^{j_1} \cdots Z_{n_\alpha}^{j_\alpha}) &= \prod_{\theta=1}^{s} \prod_{\chi=1}^{v_s} \mathbf{E}((Z_{m_\theta}^{j_{\theta,\chi}})^{u_{\theta,\chi}} \\ &= \mathbf{E}(Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_s}(\mathbf{j}_s, \mathbf{u}_s, v_s)).\end{aligned}$$

Lemma 3.2. *For λ small enough*

$$\begin{aligned}&\mathbf{E}(Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_s}(\mathbf{j}_s, \mathbf{u}_s, v_s)) \\ &\quad \leq (C\lambda)^{m_s} K^{m_s - 1} i^{\alpha d} \Phi(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots \Phi(\mathbf{j}_s, \mathbf{u}_s, v_s).\end{aligned} \tag{27}$$

Proof. We prove the lemma by induction in s. Note that the case $s = 1$ was studied in Lemma 3.1. We have

$$\begin{aligned}
&\mathbf{E}(Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_s}(\mathbf{j}_s, \mathbf{u}_s, v_s)) \\
&\quad = \mathbf{E}[(Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_{s-1}}(\mathbf{j}_{s-1}, \mathbf{u}_{s-1}, v_{s-1}) \qquad (28)\\
&\qquad \times \mathbf{E}(Z_{m_s}(\mathbf{j}_s, \mathbf{u}_s, v_s) \mid \mathcal{F}_{m_s-1})] \\
&\quad \le C\lambda\Phi(\mathbf{j}_s, \mathbf{u}_s, v_s)M(\alpha)\mathbf{E}[Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_{s-1}}(\mathbf{j}_{s-1}, \mathbf{u}_{s-1}, v_{s-1}) \qquad (29)\\
&\qquad \times \sum_{(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)} \varphi(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) Z_{m_s-1}(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)] \\
&\quad = C\lambda M(\alpha)\Phi(\mathbf{j}_s, \mathbf{u}_s, v_s)\mathbf{E}[\mathbf{E}(Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_{s-1}}(\mathbf{j}_{s-1}, \mathbf{u}_{s-1}, v_{s-1}) \qquad (30)\\
&\qquad \times \sum_{(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)} \varphi(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) Z_{m_s-1}(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) \mid \mathcal{F}_{m_s-2})] \\
&\quad = C\lambda M(\alpha)\Phi(\mathbf{j}_s, \mathbf{u}_s, v_s)\mathbf{E}[Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_{s-1}}(\mathbf{j}_{s-1}, \mathbf{u}_{s-1}, v_{s-1}) \qquad (31)\\
&\qquad \times \sum_{(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)} \varphi(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) \mathbf{E}(Z_{m_s-1}(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) \mid \mathcal{F}_{m_s-2})] \\
&\quad \le C\lambda M(\alpha)\Phi(\mathbf{j}_s, \mathbf{u}_s, v_s)\mathbf{E}[Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_{s-1}}(\mathbf{j}_{s-1}, \mathbf{u}_{s-1}, v_{s-1}) \qquad (32)\\
&\qquad \times \sum_{(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)} \varphi(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) \Phi(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s) \\
&\qquad \times C\lambda M(\alpha) \sum_{(\mathbf{t}_2, \mathbf{w}_2, h_2; \gamma_s)} \varphi(\mathbf{t}_2, \mathbf{w}_2, h_2; \gamma_s) Z_{m_s-2}(\mathbf{t}_2, \mathbf{w}_2, h_2; \gamma_s)] \\
&\quad \le (C\lambda)^2 (M(\alpha))^2 \Phi(\mathbf{j}_s, \mathbf{u}_s, v_s)\mathbf{E}[Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_{s-1}}(\mathbf{j}_{s-1}, \mathbf{u}_{s-1}, v_{s-1}) \\
&\qquad \times \sum_{(\mathbf{t}_2, \mathbf{w}_2, h_2; \gamma_s)} \varphi(\mathbf{t}_2, \mathbf{w}_2, h_2; \gamma_s) Z_{m_s-2}(\mathbf{t}_2, \mathbf{w}_2, h_2; \gamma_s)] \\
&\qquad \times \sum_{(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)} \Phi(\mathbf{t}_1, 2\mathbf{w}_1, h_1).
\end{aligned}$$

The passage from (28) to (29) is justified by (23) together with the remark that

$$\mathbf{E}(Z_{n-1}(\mathbf{t}, \mathbf{w}, h) \mid \mathcal{F}_{n-1}) = Z_{n-1}(\mathbf{t}, \mathbf{w}, h)$$

(recall that $\mu^{(n-1)}_{i,\,(\mathbf{t},\mathbf{w},h)} = \mathbf{E}Z_{n-1}(\mathbf{t}, \mathbf{w}, h)$). As $\gamma_s \le \alpha$, we have (cf. (25) in the proof of Lemma 3.1)

$$\sum_{(\mathbf{t}_1, \mathbf{w}_1, h_1; \gamma_s)} \Phi(\mathbf{t}_1, 2\mathbf{w}_1, h_1) \le L(\alpha) < \infty.$$

Continuing in this way, we get that (to save the space, abbreviate $N_s := m_s - m_{s-1}$)

$$\mathbf{E}(Z_{m_1}(\mathbf{j}_1, \mathbf{u}_1, v_1) \cdots Z_{m_s}(\mathbf{j}_s, \mathbf{u}_s, v_s))$$

$$
\begin{aligned}
\leq\ & (C\lambda)^{N_s}(M(\alpha))^{N_s}(L(\alpha))^{N_s-1}\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s)\mathbf{E}\Big[\prod_{\beta=1}^{s-1} Z_{m_\beta}(\mathbf{j}_\beta,\mathbf{u}_\beta,v_\beta)\\
& \times \sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} \varphi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s) Z_{m_{s-1}}(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\Big]\\
\leq\ & \sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} (C\lambda)^{N_s}(M(\alpha))^{N_s}(L(\alpha))^{N_s-1}\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s)\\
& \times\mathbf{E}\Big[\Big(\prod_{\beta=1}^{s-1} Z_{m_\beta}(\mathbf{j}_\beta,\mathbf{u}_\beta,v_\beta)\Big) Z_{m_{s-1}}\Big((\mathbf{j}_{s-1},\mathbf{u}_{s-1},v_{s-1})+(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\Big)\Big]\\
& \times\varphi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\\
\leq\ & \sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} (C\lambda)^{N_s}(M(\alpha))^{N_s}(L(\alpha))^{N_s-1}\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s)\\
& \times(C\lambda)^{m_{s-1}}K^{m_{s-1}-1}i^{\alpha d}\Phi(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots\Phi(\mathbf{j}_{s-2},\mathbf{u}_{s-2},v_{s-2})\\
& \times\Phi\Big((\mathbf{j}_{s-1},\mathbf{u}_{s-1},v_{s-1})+(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\Big)\varphi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\\
=\ & \sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} (C\lambda)^{N_s}(M(\alpha))^{N_s}(L(\alpha))^{N_s-1}\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s)\\
& \times(C\lambda)^{m_{s-1}}K^{m_{s-1}-1}i^{\alpha d}\Phi(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots\Phi(\mathbf{j}_{s-2},\mathbf{u}_{s-2},v_{s-2})\\
& \times\Phi(\mathbf{j}_{s-1},\mathbf{u}_{s-1},v_{s-1})\Phi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\varphi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\\
\leq\ & (C\lambda)^{m_s}(M(\alpha))^{N_s}(L(\alpha))^{N_s-1}K^{m_{s-1}-1}i^{\alpha d}\\
& \times\Phi(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s)\\
& \times\sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} \varphi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\Phi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\\
\leq\ & (C\lambda)^{m_s}K^{m_s-1}i^{\alpha d}\Phi(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s).
\end{aligned}
$$

Here we use the induction assumption, the multiplicativity of $Z_k(\cdot,\cdot,\cdot)$ and $\Phi(\cdot,\cdot,\cdot)$, and the fact that, as $\gamma_s\leq\alpha$,

$$
\begin{aligned}
&\sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} \varphi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\Phi(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)\\
&\qquad\qquad = \sum_{(\mathbf{t}_{N_s},\mathbf{w}_{N_s},h_{N_s};\gamma_s)} \Phi(\mathbf{t}_{N_s},2\mathbf{w}_{N_s},h_{N_s})\leq L(\alpha).
\end{aligned}
$$

As before, $K:=M(\alpha)L(\alpha)$. Thus, Lemma 3.2 is completely proved. □

Since $\|U^{(0)}(0)\|\leq\sum_{n=1}^\infty\sum_{j=1}^\infty 2jZ_n^j$, now it only rests to estimate, with the help of Lemma 3.2,

$$
\begin{aligned}
\mathbf{E}\Big(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} jZ_n^j\Big)^{\alpha} &\leq \alpha! \sum_{\substack{n_1\leq\ldots\leq n_\alpha\\ j_1,\ldots,j_\alpha}} j_1\cdots j_\alpha \mathbf{E}(Z_{n_1}^{j_1}\cdots Z_{n_\alpha}^{j_\alpha}) \qquad (33)\\
&\leq C' \sum_{\substack{m_1<\ldots<m_s\\ 1\leq s\leq\alpha}} \sum_{\substack{(\mathbf{j}_l,\mathbf{u}_l,v_l)\\ l=1,\ldots,s}} j_{l,1}^{u_{l,1}}\cdots j_{l,v_l}^{u_{l,v_l}}\\
&\quad \times \mathbf{E}(Z_{m_1}(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots Z_{m_s}(\mathbf{j}_s,\mathbf{u}_s,v_s))\\
&\leq C' i^{\alpha d} \sum_{\substack{m_1<\ldots<m_s\\ 1\leq s\leq\alpha}} (C\lambda)^{m_s} K^{m_s-1}\\
&\quad \times \sum_{\substack{(\mathbf{j}_l,\mathbf{u}_l,v_l)\\ l=1,\ldots,s}} j_{l,1}^{u_{l,1}}\cdots j_{l,v_l}^{u_{l,v_l}} \Phi(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s)\\
&\leq \frac{C''}{K} \sum_{\substack{m_1<\ldots<m_s\\ 1\leq s\leq\alpha}} (CK\lambda)^{m_s}\\
&\leq \frac{C''}{K} \sum_{m=1}^{\infty} \alpha m^{\alpha}(CK\lambda)^m < \infty
\end{aligned}
$$

for $\lambda < (CK)^{-1}$, where C' and C'' are positive numbers which depend only on α. Here we used the fact that

$$
\sum_{\substack{(\mathbf{j}_l,\mathbf{u}_l,v_l)\\ l=1,\ldots,s}} j_{l,1}^{u_{l,1}}\cdots j_{l,v_l}^{u_{l,v_l}} \Phi(\mathbf{j}_1,\mathbf{u}_1,v_1)\cdots\Phi(\mathbf{j}_s,\mathbf{u}_s,v_s) < \infty,
$$

which can be easily proved analogously to (25). Thus, the proof of Theorem 1.2 is completed. □

Remark. Let $N(U^{(0)}(0))$ be the number of points in the cluster $U^{(0)}(0)$ and $|U^{(0)}(0)|$ be the volume covered by the cluster. Then, in the same way it can be proved that if $\mathbf{E}\rho^{2\alpha d} < \infty$, then for λ small enough we have $\mathbf{E}(N(U^{(0)}(0)))^{\alpha} < \infty$ and $\mathbf{E}(|U^{(0)}(0)|)^{\alpha} < \infty$ (substitute jZ_n^j in (33) by Z_n^j in the former, and $j^d Z_n^j$ in the latter case.)

References

[1] I. Benjamini and O. Schramm, Exceptional planes of percolation. *Probab. Theory Related Fields* **111**(4): (1998), 551–564.

[2] J.T. Chayes, L. Chayes and R. Durrett, Connectivity properties of Mandelbrot's percolation process. *Probab. Theory Relat. Fields* **77:** (1988), 307–324.

[3] L. Chayes, R. Pemantle and Y. Peres, No directed fractal percolation in zero area. *J. Statist. Phys.* **88**(5/6): (1997), 1353–1362.

[4] K.J. Falconer and G.R. Grimmett, On the geometry of random Cantor sets and fractal percolation. *J. Theoret. Probab.* **5**(3): (1992), 465–485.

[5] A.O. Golosov, S.A. Molchanov and A.Ya. Reznikova, Continuous percolation models in fracture theory. *Vychisl. Seismol.* **20:** (1986).

[6] J.C. Gupta and B.V. Rao, Van den Berg-Kesten inequality for the Poisson Boolean model for continuum percolation. *Sankhyā Ser. A* **61**(3): (1999), 337–346.

[7] F.A. Haight, *Handbook of the Poisson distribution.* Publications in Operations Research (11), John Wiley & Sons, Inc., New York–London–Sydney, (1967).

[8] P. Hall, On continuum percolation. *Ann. Probab.* **13**(4): (1985), 1250–1266.

[9] T.M. Liggett, R.H. Schonmann and A.M. Stacey, Domination by product measures. *Ann. Probab.* **25**(1): (1997), 71–95.

[10] B. Mandelbrot, Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier. *J. Fluid Mech.* **62:** (1974), 331–358.

[11] R. Meester and R. Roy, *Continuum percolation.* Cambridge tracts in Mathematics, **119:** (1996). Cambridge University Press, Cambridge.

[12] R. Meester, R. Roy and A. Sarkar, Non-universality and continuity of the critical covered volume fraction. *J. Statist. Phys.* **75:** (1994), 123–134.

[13] M.V. Menshikov, S.A. Molchanov and A.F. Sidorenko, Percolation theory and some applications. *Itogi Nauki i Tekhniki (Probability Theory, Mathematical Statistics, Theoretical Cybernetics)* **24:** (1986), 53–110. English translation in: *J. Soviet Math.* **42:** (1988), 1766–1810.

[14] M.V. Menshikov, S.Yu. Popov and M. Vachkovskaia, On the connectivity properties of the complementary set in fractal percolation models. *Probab. Theory Relat. Fields.* **119**(2): (2001), 176–186.

[15] M.V. Menshikov, S.Yu. Popov and M. Vachkovskaia, Multiscale percolation on k-symmetric mosaic. *Statist. Probab. Letters* **52**(1): (2001), 79–84.

[16] M.V. Menshikov and A.F. Sidorenko, Coincidence of critical points for Poisson percolation models. *Theory Probab. Appl.* **32:** (1987), 603–606.

[17] M.E. Orzechowski, A lower bound on the box-counting dimension of crossings in fractal percolation. *Stochastic Process. Appl.* **74**(1): (1998), 53–65.

[18] S.Yu. Popov and M. Vachkovskaia, A note on percolation of Poisson sticks. *Braz. J. Probab. Stat.* **16:** (2002), 59–67.

[19] A. Sarkar, Co-existence of the occupied and vacant phase in Boolean models in three or more dimensions. *Adv. Appl. Probab.* **29**(4): (1997), 878–889.

[20] A. Sarkar, Finite clusters in high density Boolean models with balls of varying sizes. *Adv. Appl. Probab.* **30**(4): (1998), 929–947.

[21] S.A. Zuev and A.F. Sidorenko, Continuous models of percolation theory I. (In Russian). *Teoret. Mat. Fiz.* **62**(1): (1985), 76–86.

[22] S.A. Zuev and A.F. Sidorenko, (1985) Continuous models of percolation theory II. (In Russian). *Teoret. Mat. Fiz.* **62**(2): (1985), 253–262.

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